

Fusion procedure for Coxeter groups of type B and complex reflection groups $G(m,1,n)$

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Abstract

A complete system of primitive pairwise orthogonal idempotents for the Coxeter groups of type B and, more generally, for the complex reflection groups $G(m, 1, n)$ is constructed by a sequence of evaluations of a rational function in several variables with values in the group ring. The evaluations correspond to the eigenvalues of the two arrays of Jucys–Murphy elements.

1. Introduction

A. Jucys [12] gave a construction of a complete system of pairwise orthogonal primitive idempotents of the group ring of the symmetric group; the construction, called now *fusion procedure*, involves a rational function in several variables and the idempotents are obtained by taking certain limiting values of this function. We refer to [1, 2, 4, 9, 15, 16, 18] for different aspects and applications of the fusion procedure for the symmetric groups. There are analogues of the fusion procedure for the Hecke algebra of type A [3, 17] and the spinor extension of the symmetric group [14, 10] (see [11] for its q -analogue).

A version of the fusion procedure for the symmetric group was given by A. Molev in [13]. Here the idempotents are obtained by consecutive evaluations of the rational function. An analogue of this fusion procedure was developed for the Hecke algebra [6], the Brauer algebra [5, 7] and the Birman–Murakami–Wenzl algebra [8].

The aim of this paper is to give a fusion procedure, in the spirit of [13], for the complex reflection groups $G(m, 1, n)$. As in [13], and later [5, 6, 7, 8], we use the Jucys–Murphy elements. They were introduced for $G(m, 1, n)$ independently in [21] and [23]. The Jucys–Murphy elements of $G(m, 1, n)$ form two arrays, j_i and \tilde{j}_i , $i = 1, \dots, n$, and their union is the maximal commutative set in $CG(m, 1, n)$,

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see [20, 21]. An irreducible representation of the group $G(m, 1, n)$ is coded by an m -tuple of partitions and the elements of the semi-normal basis correspond to standard m -tuples of tableaux; the eigenvalues of j_i carry information about the “position” - the place of a tableau in the m -tuple - while the eigenvalues of \tilde{j}_i are related to the classical contents of nodes. In the work [20] both sets appeared as classical limits of simple expressions involving the single set of the Jucys–Murphy elements of the cyclotomic Hecke algebra, the flat deformation of $\mathbb{C}G(m, 1, n)$. By the maximality, all diagonal matrix units of $\mathbb{C}G(m, 1, n)$ can be expressed in terms of the Jucys–Murphy elements j_i and \tilde{j}_i , $i = 1, \dots, n$. Then we translate this expression as a fusion procedure: any diagonal matrix unit can be obtained by a sequence of evaluations of a certain rational function with values in $\mathbb{C}G(m, 1, n)$. The arrays j_i and \tilde{j}_i play different roles: the positions can be evaluated simultaneously while the contents should then be evaluated consequently from 1 to n .

The group $G(1, 1, n)$ is isomorphic to the symmetric group S_n and our fusion procedure for $m = 1$ reproduces the fusion procedure of [13].

The group $G(2, 1, n)$ is isomorphic to the hyperoctahedral group B_n , the Coxeter group of type B. Thus, in particular, we obtain a fusion procedure for the Coxeter group of type B and a description of a complete set of pairwise orthogonal primitive idempotents of B_n in terms of a single rational function with values in the group algebra $\mathbb{C}B_n$.

For the clarity of the exposition, we first describe the fusion procedure for the Coxeter group B_n . This is done not only for aesthetic reasons: the rational function with values in $\mathbb{C}B_n$ leading to the complete set of idempotents can be viewed as a word for the longest element of B_n in which certain entries are “Baxterized”, similarly to the rational function for the type A. Although $G(m, 1, n)$ is not a Coxeter group for $m > 2$ and the notion of length of an element is not defined, there is an analogue of the longest element: it is longest with respect to the normal form of [20] (the classical limit of the normal form for the cyclotomic Hecke algebra $H(m, 1, n)$ [19]); the rational function with values in $\mathbb{C}G(m, 1, n)$ leading to the complete set of idempotents can again be viewed as a word for the longest element with certain entries Baxterized. We stress that the groups $G(m, 1, n)$ admit a fusion procedure for any positive integer m , and our construction is uniform for all m .

The paper is organized as follows. Section 2 contains necessary definitions and notations about the groups B_n and their representations. The Jucys–Murphy elements for B_n were defined in [21, 22, 23]; we describe the diagonal matrix units in terms of them. In Section 3 we prove the Theorem 2 which gives the fusion procedure for the groups B_n . In Section 4, in the Theorem 6, we extend the results to the general case of the groups $G(m, 1, n)$. As the proofs work mainly along the same lines as for B_n , we only indicate the necessary modifications.

2. Idempotents and Jucys–Murphy elements of the group B_n

2.1 Definitions

The Coxeter group A_n of type A (the symmetric group on $n + 1$ letters) is generated by the elements s_1, \dots, s_n with the defining relations:

$$\begin{aligned} s_i^2 &= 1 && \text{for } i = 1, \dots, n, \\ s_i s_{i+1} s_i &= s_{i+1} s_i s_{i+1} && \text{for } i = 1, \dots, n - 1, \\ s_i s_j &= s_j s_i && \text{for } i, j = 1, \dots, n \text{ such that } |i - j| > 1. \end{aligned} \tag{1}$$

The Coxeter group B_{n+1} of type B (also called the hyperoctahedral group) is generated by the elements s_1, \dots, s_n and t with the defining relations (1),

$$\begin{aligned} ts_1ts_1 &= s_1ts_1t, \\ s_it &= ts_i \quad \text{for } i = 2, \dots, n \end{aligned} \quad (2)$$

and

$$t^2 = 1. \quad (3)$$

For any $i = 1, \dots, n$, set

$$s_i(p, p', a, a') := s_i + \frac{\delta_{p,p'}}{a - a'}, \quad (4)$$

where $\delta_{p,p'}$ is the Kronecker delta. For $p = p'$ the elements (4) are called Baxterized elements; the parameters a and a' are referred to as spectral parameters. We also define

$$\mathbf{t}(p) := \frac{1}{2}(1 + pt). \quad (5)$$

The following relation is satisfied and will be used later:

$$s_i(p, p', a, a')s_i(p', p, a', a) = \frac{(a - a')^2 - \delta_{p,p'}}{(a - a')^2} \quad \text{for } i = 1, \dots, n. \quad (6)$$

Define the elements j_i , $i = 1, \dots, n + 1$, and \tilde{j}_i , $i = 1, \dots, n + 1$, of the group algebra $\mathbb{C}B_{n+1}$ by the following initial conditions and recursions:

$$j_1 = t, \quad j_{i+1} = s_i j_i s_i \quad \text{and} \quad \tilde{j}_1 = 0, \quad \tilde{j}_{i+1} = s_i \tilde{j}_i s_i + \frac{1}{2}(s_i + j_i s_i j_i). \quad (7)$$

The elements j_i and \tilde{j}_i are analogues, for the group B_{n+1} , of the Jucys–Murphy elements. The elements j_i , $i = 1, \dots, n + 1$, and \tilde{j}_i , $i = 1, \dots, n + 1$, form a maximal commutative set in $\mathbb{C}B_{n+1}$, see [20, 21, 22]; in addition, j_i and \tilde{j}_i commute with all generators s_k , except s_i and s_{i-1} :

$$j_i s_k = s_k j_i \quad \text{and} \quad \tilde{j}_i s_k = s_k \tilde{j}_i \quad \text{if } k \neq i - 1, i. \quad (8)$$

Let $\lambda \vdash n + 1$ be a partition of length $n + 1$, that is, $\lambda = (\lambda_1, \dots, \lambda_k)$, where λ_j , $j = 1, \dots, k$, are positive integers, $\lambda_1 \geq \lambda_2 \geq \dots \geq \lambda_k$ and $n + 1 = \lambda_1 + \dots + \lambda_k$. We identify partitions with their Young diagrams: the Young diagram of λ is a left-justified array of rows of nodes containing λ_j nodes in the j -th row, $j = 1, \dots, k$; the rows are numbered from top to bottom.

A 2-partition, or a Young 2-diagram, of length $n + 1$ is a pair of partitions such that the sum of their lengths equals $n + 1$. A 2-node $\alpha^{(2)}$ is a pair (α, k) consisting of a usual node α and an integer $k = 1, 2$. The integer k will be called *position* of the 2-node. A set of 2-nodes can be equivalently described by an ordered pair of sets of nodes (the integer k of a 2-node (α, k) indicates to which set the node α belongs). A 2-partition $\lambda^{(2)}$ is a set of 2-nodes such that the subset consisting of the 2-nodes having position p is a usual partition, $p = 1, 2$.

For a 2-node $\alpha^{(2)} = (\alpha, k)$ lying in the line x and the column y of the k -th diagram, we denote by $c(\alpha^{(2)})$ the classical content of the node α , $c(\alpha^{(2)}) := c(\alpha) = y - x$. Let $\{\xi_1, \xi_2\}$ be the set of distinct square roots of unity, ordered arbitrarily; we define also $p(\alpha^{(2)}) := \xi_k$.

For a 2-partition $\lambda^{(2)}$, a 2-node $\alpha^{(2)}$ of $\lambda^{(2)}$ is called *removable* if the set of 2-nodes obtained from $\lambda^{(2)}$ by removing $\alpha^{(2)}$ is still a 2-partition. A 2-node $\beta^{(2)}$ not in $\lambda^{(2)}$ is called *addable* if the set of 2-nodes obtained from $\lambda^{(2)}$ by adding $\beta^{(2)}$ is still a 2-partition. For a 2-partition $\lambda^{(2)}$, we denote by $\mathcal{E}_-(\lambda^{(2)})$ the set of removable 2-nodes of $\lambda^{(2)}$ and by $\mathcal{E}_+(\lambda^{(2)})$ the set of addable 2-nodes of $\lambda^{(2)}$.

Let $\lambda^{(2)}$ be a 2-partition of length $n + 1$. A standard 2-tableau of shape $\lambda^{(2)}$ is obtained by placing the numbers $1, \dots, n + 1$ in the 2-nodes of the diagrams of $\lambda^{(2)}$ in such a way that the numbers in the nodes ascend along rows and columns in every diagram. For a standard 2-tableau \mathcal{T} of shape $\lambda^{(2)}$ let $\alpha_i^{(2)}$ be the 2-node of \mathcal{T} with number i , $i = 1, \dots, n + 1$; we set $c(\mathcal{T}|i) := c(\alpha_i^{(2)})$ and $p(\mathcal{T}|i) := p(\alpha_i^{(2)})$.

The hook of a node α of a partition ν is the set of nodes of ν consisting of the node α and the nodes which lie either under α in the same column or to the right of α in the same row; the hook length $h_\nu(\alpha)$ of α is the cardinality of the hook of α . We extend this definition to 2-nodes. For a 2-node $\alpha^{(2)} = (\alpha, k)$ of a 2-partition $\nu^{(2)}$, the hook length of $\alpha^{(2)}$ in $\nu^{(2)}$, which we denote by $h_{\nu^{(2)}}(\alpha^{(2)})$, is the hook length of the node α in the k -th partition of $\nu^{(2)}$. Let

$$f_{\nu^{(2)}} := \left(\prod_{\alpha^{(2)} \in \nu^{(2)}} h_{\nu^{(2)}}(\alpha^{(2)}) \right)^{-1}. \quad (9)$$

2.2 Idempotents of the group B_{n+1}

The representation theory of the Coxeter groups of type B was developed by A. Young [24], see also [20, 21, 22]. The irreducible representations of B_{n+1} are in bijection with the 2-partitions of length $n + 1$. The elements of the semi-normal basis of the irreducible representation of B_{n+1} corresponding to the 2-partition $\lambda^{(2)}$ are parameterized by the standard 2-tableaux of shape $\lambda^{(2)}$. For a standard 2-tableau \mathcal{T} , we denote by $E_{\mathcal{T}}$ the primitive idempotent of B_{n+1} corresponding to \mathcal{T} . The Jucys–Murphy elements are diagonal in the semi-normal basis; moreover, we have, for any $i = 1, \dots, n + 1$,

$$j_i E_{\mathcal{T}} = E_{\mathcal{T}} j_i = p_i E_{\mathcal{T}} \quad \text{and} \quad \tilde{j}_i E_{\mathcal{T}} = E_{\mathcal{T}} \tilde{j}_i = c_i E_{\mathcal{T}}. \quad (10)$$

Here we set $p_i := p(\mathcal{T}|i)$ and $c_i := c(\mathcal{T}|i)$ for all $i = 1, \dots, n + 1$ for brevity. Due to the maximality of the commutative set $\{j_i, \tilde{j}_i\}_{i=1, \dots, n+1}$ of Jucys–Murphy elements, the idempotent $E_{\mathcal{T}}$ can be expressed in terms of j_i, \tilde{j}_i , $i = 1, \dots, n + 1$. Let $\alpha^{(2)}$ be the 2-node of \mathcal{T} with the number $n + 1$. As the tableau \mathcal{T} is standard, the 2-node $\alpha^{(2)}$ of $\lambda^{(2)}$ is removable. Let \mathcal{U} be the standard 2-tableau obtained from \mathcal{T} by removing the 2-node $\alpha^{(2)}$ and let $\mu^{(2)}$ be the shape of \mathcal{U} . The inductive formula for $E_{\mathcal{T}}$ in terms of the Jucys–Murphy elements reads:

$$E_{\mathcal{T}} = E_{\mathcal{U}} \prod_{\substack{\beta^{(2)} \in \mathcal{E}_+(\mu^{(2)}) \\ c(\beta^{(2)}) \neq c(\alpha^{(2)})}} \frac{\tilde{j}_{n+1} - c(\beta^{(2)})}{c(\alpha^{(2)}) - c(\beta^{(2)})} \prod_{\substack{\beta^{(2)} \in \mathcal{E}_+(\mu^{(2)}) \\ p(\beta^{(2)}) \neq p(\alpha^{(2)})}} \frac{j_{n+1} - p(\beta^{(2)})}{p(\alpha^{(2)}) - p(\beta^{(2)})}. \quad (11)$$

Note that the second product in the right hand side of (11) contains only one term (it will not be so for the cyclotomic groups $G(m, 1, n)$ with $m > 2$). We have $B_0 \cong \mathbb{C}$ and $E_{\mathcal{U}_0} = 1$ for the (unique) 2-tableau \mathcal{U}_0 of length 0.

Let $\{\mathcal{T}_1, \dots, \mathcal{T}_k\}$ be the set of pairwise different standard 2-tableaux that can be obtained from \mathcal{U} by adding a 2-node with the number $n + 1$. The following formula:

$$E_{\mathcal{U}} = \sum_{i=1}^k E_{\mathcal{T}_i},$$

together with (10) implies that the rational function

$$E_{\mathcal{U}} \frac{u - c_{n+1}}{u - \tilde{j}_{n+1}} \frac{v - p_{n+1}}{v - j_{n+1}}$$

is non-singular at $u = c_{n+1}$ and $v = p_{n+1}$, and, moreover,

$$E_{\mathcal{U}} \frac{u - c_{n+1}}{u - \tilde{j}_{n+1}} \frac{v - p_{n+1}}{v - j_{n+1}} \Big|_{\substack{u = c_{n+1} \\ v = p_{n+1}}} = E_{\mathcal{T}}. \quad (12)$$

Since j_{n+1} takes values ± 1 , the rational function $\frac{v - p_{n+1}}{v - j_{n+1}}$ is non-singular at $v = p_{n+1}$ and

$$\frac{v - p_{n+1}}{v - j_{n+1}} \Big|_{v = p_{n+1}} = \frac{1}{2}(1 + p_{n+1}j_{n+1}). \quad (13)$$

For the clarity of the calculations in the sequel, we define, generalizing (5),

$$\mathbf{j}_i(p) := \frac{1}{2}(1 + pj_i) \quad \text{for } i = 1, \dots, n + 1. \quad (14)$$

Combining (12) and (13), we obtain the following formula for the idempotent $E_{\mathcal{T}}$:

$$E_{\mathcal{T}} = E_{\mathcal{U}} \mathbf{j}_{n+1}(p_{n+1}) \frac{u - c_{n+1}}{u - \tilde{j}_{n+1}} \Big|_{u = c_{n+1}}. \quad (15)$$

3. Fusion formula for idempotents of B_{n+1}

We start with the following Lemma which will be useful in the sequel.

Lemma 1. *For any integer l , $1 \leq l \leq n$, we have*

- (i) $\tilde{j}_{n+1} = s_n s_{n-1} \dots s_l \tilde{j}_l s_l \dots s_{n-1} s_n + \frac{1}{2} \sum_{i=l}^n s_n \dots s_{i+1} s_i s_{i+1} \dots s_n (1 + j_{n+1} j_i),$
- (ii) $\mathbf{j}_l(p) s_l \dots s_{n-1} s_n \tilde{j}_{n+1} = \mathbf{j}_l(p) \tilde{j}_l s_l \dots s_{n-1} s_n + \frac{1}{2} \sum_{i=l}^n s_l s_{l+1} \dots s_{i-1} \cdot s_{i+1} \dots s_{n-1} s_n \mathbf{j}_i(p) (1 + j_{n+1} j_i);$

the product $s_l s_{l+1} \dots s_{i-1}$ in the right-hand-side of (ii) is understood to be equal to 1 if $i = l$.

Proof. We prove (i) by induction on $n - l$. The basis of the induction is, for $l = n$, the formula $\tilde{j}_{n+1} = s_n \tilde{j}_n s_n + \frac{1}{2}(s_n + s_n j_{n+1} j_n)$ which follows from the definition (7) of the Jucys–Murphy elements, namely from $\tilde{j}_{n+1} = s_n \tilde{j}_n s_n + \frac{1}{2}(s_n + j_n s_n j_n)$ and $j_n s_n = s_n j_{n+1}$. Now assume that

$$\tilde{j}_{n+1} = s_n s_{n-1} \dots s_{l+1} \tilde{j}_{l+1} s_{l+1} \dots s_{n-1} s_n + \frac{1}{2} \sum_{i=l+1}^n s_n \dots s_{i+1} s_i s_{i+1} \dots s_n (1 + j_{n+1} j_i),$$

and replace \tilde{j}_{l+1} by $s_l \tilde{j}_l s_l + \frac{1}{2}(s_l + j_l s_l j_l)$. One obtains the assertion (i) using that $j_l s_{l+1} \dots s_{n-1} s_n = s_{l+1} \dots s_{n-1} s_n j_l$ and that $j_l s_l s_{l+1} \dots s_{n-1} s_n = s_l s_{l+1} \dots s_{n-1} s_n j_{n+1}$.

Using (i), we find

$$\mathbf{j}_l(p) s_l \dots s_{n-1} s_n \tilde{j}_{n+1} = \mathbf{j}_l(p) \left(\tilde{j}_l s_l \dots s_{n-1} s_n + \frac{1}{2} \sum_{i=l}^n s_l s_{l+1} \dots s_{i-1} \cdot s_{i+1} \dots s_{n-1} s_n (1 + j_{n+1} j_i) \right),$$

and (ii) follows since $j_l s_l s_{l+1} \dots s_{i-1} = s_l s_{l+1} \dots s_{i-1} j_i$ and j_i commutes with $s_{i+1} \dots s_{n-1} s_n$. \square

Let $\phi_1(v, u) := \mathbf{t}(v)$; for $k = 1, \dots, n$ define

$$\begin{aligned} \phi_{k+1}(v_1, \dots, v_k, v, u_1, \dots, u_k, u) &:= \mathbf{s}_k(v, v_k, u, u_k) \phi_k(v_1, \dots, v_{k-1}, v, u_1, \dots, u_{k-1}, u) s_k \\ &= \mathbf{s}_k(v, v_k, u, u_k) \mathbf{s}_{k-1}(v, v_{k-1}, u, u_{k-1}) \dots \mathbf{s}_1(v, v_1, u, u_1) \mathbf{t}(v) s_1 \dots s_{k-1} s_k. \end{aligned} \quad (16)$$

Define the following rational function with values in the group ring of \mathbf{B}_{n+1} :

$$\Phi(v_1, \dots, v_{n+1}, u_1, \dots, u_{n+1}) := \prod_{k=0, \dots, n}^{\leftarrow} \phi_{k+1}(v_1, \dots, v_k, v_{k+1}, u_1, \dots, u_k, u_{k+1}) ; \quad (17)$$

the arrow over \prod indicates that the (non-commuting) factors are taken in the descending order.

Let $\lambda^{(2)}$ be a 2-partition of length $n + 1$ and \mathcal{T} a standard 2-tableau of shape $\lambda^{(2)}$. For $i = 1, \dots, n + 1$, set $p_i := p(\mathcal{T}|i)$ and $c_i := c(\mathcal{T}|i)$.

Theorem 2. *The idempotent $E_{\mathcal{T}}$ corresponding to the standard 2-tableau \mathcal{T} of shape $\lambda^{(2)}$ can be obtained by the following consecutive evaluations*

$$E_{\mathcal{T}} = f_{\lambda^{(2)}} \Phi(v_1, \dots, v_{n+1}, u_1, \dots, u_{n+1}) \Big|_{v_i=p_i, i=1, \dots, n+1} \Big|_{u_1=c_1} \dots \Big|_{u_n=c_n} \Big|_{u_{n+1}=c_{n+1}}. \quad (18)$$

Proof. Define

$$F_{\mathcal{T}}(u) := \frac{u - c_{n+1}}{u} \prod_{i=1}^n \frac{(u - c_i)^2}{(u - c_i)^2 - \delta_{p_i, p_{n+1}}}. \quad (19)$$

Let \mathcal{U} be the standard 2-tableau obtained from \mathcal{T} by removing the 2-node with the number $n + 1$ and let $\mu^{(2)}$ be the shape of \mathcal{U} .

Proposition 3. *We have*

$$F_{\mathcal{T}}(u) \phi_{n+1}(p_1, \dots, p_n, p_{n+1}, c_1, \dots, c_n, u) E_{\mathcal{U}} = \frac{u - c_{n+1}}{u - \tilde{j}_{n+1}} \mathbf{j}_{n+1}(p_{n+1}) E_{\mathcal{U}}. \quad (20)$$

Proof. We prove (20) by induction on n . As $c_1 = 0$ and $\tilde{j}_1 = 0$, the basis of induction for $n = 0$ is the formula $\mathbf{t}(p_1) = \mathbf{j}_1(p_1)$ which is satisfied by definition, see (5) and (14).

If $p_{n+1} \neq p_i$, $i = 2, \dots, n$, then fix $l = 1$. Otherwise fix l such that $p_{n+1} = p_l$ and $p_{n+1} \neq p_i$, $i = l+1, \dots, n$.

Define \mathcal{V} to be the standard 2-tableau obtained from \mathcal{U} by removing the 2-nodes containing the numbers $l+1, \dots, n$ and \mathcal{W} to be the standard 2-tableau obtained from \mathcal{V} by removing the 2-node with the number l . We will use that $E_{\mathcal{W}}E_{\mathcal{U}} = E_{\mathcal{U}}$ and that $E_{\mathcal{W}}$ commutes with s_i , for $i = l, l+1, \dots, n$. We rewrite the left-hand side of (20) as

$$F_{\mathcal{T}}(u)s_n \dots s_{l+1}\mathbf{s}_l(p_{n+1}, p_l, u, c_l) \cdot \phi_l(p_1, \dots, p_{l-1}, p_{n+1}, c_1, \dots, c_{l-1}, u)E_{\mathcal{W}} \cdot s_l s_{l+1} \dots s_n E_{\mathcal{U}}.$$

If $l > 1$ then we use the induction hypothesis, namely

$$\phi_l(p_1, \dots, p_{l-1}, p_l, c_1, \dots, c_{l-1}, u)E_{\mathcal{W}} = (F_{\mathcal{V}}(u))^{-1} \frac{u - c_l}{u - \tilde{j}_l} \mathbf{j}_l(p_l)E_{\mathcal{W}},$$

and we notice that $p_{n+1} = p_l$.

If $l = 1$ then $E_{\mathcal{W}} = 1$, $F_{\mathcal{V}}(u) = 1$ and, by definition, $\phi_1(p_{n+1}, u)$ is equal to $\mathbf{j}_1(p_{n+1})$. Thus, in both situations (for $l = 1$ and for $l > 1$), we obtain for the left-hand-side of (20):

$$F_{\mathcal{T}}(u)(F_{\mathcal{V}}(u))^{-1} s_n \dots s_{l+1}\mathbf{s}_l(p_{n+1}, p_l, u, c_l) \frac{u - c_l}{u - \tilde{j}_l} \mathbf{j}_l(p_{n+1}) s_l s_{l+1} \dots s_n E_{\mathcal{U}}.$$

Therefore, the equality (20) is equivalent to

$$\begin{aligned} & F_{\mathcal{T}}(u)(F_{\mathcal{V}}(u))^{-1} (u - c_l) \mathbf{j}_l(p_{n+1}) s_l s_{l+1} \dots s_n (u - \tilde{j}_{n+1}) E_{\mathcal{U}} \\ &= \frac{(u - c_{n+1})(u - c_l)^2}{(u - c_l)^2 - \delta_{p_l, p_{n+1}}} (u - \tilde{j}_l) \mathbf{s}_l(p_l, p_{n+1}, c_l, u) s_{l+1} \dots s_n \mathbf{j}_{n+1}(p_{n+1}) E_{\mathcal{U}}, \end{aligned} \quad (21)$$

where, in moving $s_n \dots s_{l+1}\mathbf{s}_l(p_{n+1}, p_l, u, c_l) \frac{1}{u - \tilde{j}_l}$ to the right-hand-side and $\frac{1}{u - \tilde{j}_{n+1}}$ to the left-hand-side, we have used that \tilde{j}_{n+1} commutes with j_{n+1} and $E_{\mathcal{U}}$, and also the formula (6) to take the inverse of $\mathbf{s}_l(p_{n+1}, p_l, u, c_l)$.

To prove the equality (21), first notice that we have

$$F_{\mathcal{T}}(u)(F_{\mathcal{V}}(u))^{-1} (u - c_l) = (u - c_{n+1}) \prod_{i=1}^n \frac{(u - c_i)^2}{(u - c_i)^2 - \delta_{p_i, p_{n+1}}} \prod_{i=1}^{l-1} \left(\frac{(u - c_i)^2}{(u - c_i)^2 - \delta_{p_i, p_l}} \right)^{-1},$$

which gives, since $p_i \neq p_{n+1}$ if $i > l$ and $p_l = p_{n+1}$ if $l > 1$,

$$F_{\mathcal{T}}(u)(F_{\mathcal{V}}(u))^{-1} (u - c_l) = \frac{(u - c_{n+1})(u - c_l)^2}{(u - c_l)^2 - \delta_{p_l, p_{n+1}}}. \quad (22)$$

So it remains to prove that

$$\mathbf{j}_l(p_{n+1}) s_l s_{l+1} \dots s_n (u - \tilde{j}_{n+1}) E_{\mathcal{U}} = (u - \tilde{j}_l) \mathbf{s}_l(p_l, p_{n+1}, c_l, u) s_{l+1} \dots s_n \mathbf{j}_{n+1}(p_{n+1}) E_{\mathcal{U}}. \quad (23)$$

Expand $s_l(p_l, p_{n+1}, c_l, u)$ in the right hand side of (23):

$$\left((u - \tilde{j}_l)s_l - \delta_{p_l, p_{n+1}} \frac{\tilde{j}_l - u}{c_l - u}\right) s_{l+1} \dots s_n \mathbf{j}_{n+1}(p_{n+1}) E_{\mathcal{U}}. \quad (24)$$

As \tilde{j}_l commutes with $s_{l+1} \dots s_n$ and j_{n+1} and $\tilde{j}_l E_{\mathcal{U}} = c_l E_{\mathcal{U}}$, we find that the expression (24) equals

$$\left((u - \tilde{j}_l)s_l s_{l+1} \dots s_n \mathbf{j}_{n+1}(p_{n+1}) - \delta_{p_l, p_{n+1}} s_{l+1} \dots s_n \mathbf{j}_{n+1}(p_{n+1})\right) E_{\mathcal{U}}.$$

Then using that $s_l s_{l+1} \dots s_n j_{n+1} = j_l s_l s_{l+1} \dots s_n$, we obtain for the right-hand-side of (23):

$$\left((u - \tilde{j}_l) \mathbf{j}_l(p_{n+1}) s_l s_{l+1} \dots s_n - \delta_{p_l, p_{n+1}} s_{l+1} \dots s_n \mathbf{j}_{n+1}(p_{n+1})\right) E_{\mathcal{U}}. \quad (25)$$

Using the Lemma 1, (ii), we write the left hand side of (23) in the form

$$\left(\mathbf{j}_l(p_{n+1})(u - \tilde{j}_l) s_l s_{l+1} \dots s_n - \frac{1}{2} \sum_{i=l}^n s_l s_{l+1} \dots s_{i-1} \cdot s_{i+1} \dots s_{n-1} s_n \mathbf{j}_i(p_{n+1})(1 + j_{n+1} j_i)\right) E_{\mathcal{U}}. \quad (26)$$

As $j_i E_{\mathcal{U}} = p_i E_{\mathcal{U}}$, $i = 1, \dots, n$, the expression (26) is equal to

$$\left(\mathbf{j}_l(p_{n+1})(u - \tilde{j}_l) s_l s_{l+1} \dots s_n - \frac{1}{2} \sum_{i=l}^n s_l s_{l+1} \dots s_{i-1} \cdot s_{i+1} \dots s_{n-1} s_n \frac{1}{2} (1 + p_i p_{n+1})(1 + p_i j_{n+1})\right) E_{\mathcal{U}}. \quad (27)$$

Since $p_k p_{n+1} = -1$, $k = l + 1, \dots, n$, we finally obtain for the left-hand-side of (23):

$$\left(\mathbf{j}_l(p_{n+1})(u - \tilde{j}_l) s_l s_{l+1} \dots s_n - \delta_{p_l, p_{n+1}} s_{l+1} \dots s_{n-1} s_n \mathbf{j}_{n+1}(p_l)\right) E_{\mathcal{U}}. \quad (28)$$

The comparison of (28) and (25) proves the equality (23). \square

Proposition 4. *The rational function $F_{\mathcal{T}}(u)$, defined by (19), is regular at $u = c_{n+1}$ and moreover*

$$F_{\mathcal{T}}(c_{n+1}) = f_{\lambda(2)}(f_{\mu(2)})^{-1}. \quad (29)$$

Proof. The Proposition will directly follow from the result, used in [13], concerning usual tableaux. Let λ be a partition of length $m + 1$, with $m \leq n$. let S be a subset of $\{1, \dots, n + 1\}$ such that S contains the number $n + 1$ and has a cardinal equal to $m + 1$. Let $\tilde{\mathcal{T}}$ be a tableau of shape λ filled with numbers belonging to S such that the numbers in the nodes are in strictly ascending orders along rows and columns in right and down directions. Let γ be the node of $\tilde{\mathcal{T}}$ with the number $n + 1$ and μ be the shape of the tableau obtained from $\tilde{\mathcal{T}}$ by removing the node γ . Define the following rational function

$$\tilde{F}_{\tilde{\mathcal{T}}}(u) = \frac{u - c(\gamma)}{u} \prod_{\alpha \in \mu} \frac{(u - c(\alpha))^2}{(u - c(\alpha) + 1)(u - c(\alpha) - 1)}. \quad (30)$$

The product in the right hand side of (30) depends only on the shape μ and one has

$$\prod_{\alpha \in \mu} \frac{(u - c(\alpha))^2}{(u - c(\alpha) + 1)(u - c(\alpha) - 1)} = u \prod_{\beta \in \mathcal{E}_-(\mu)} (u - c(\beta)) \prod_{\alpha \in \mathcal{E}_+(\mu)} (u - c(\alpha))^{-1},$$

where $\mathcal{E}_-(\mu)$ (respectively, $\mathcal{E}_+(\mu)$) is the set of removable (respectively, addable) nodes of μ . Therefore the rational function $\tilde{F}_{\tilde{\mathcal{T}}}(u)$ is regular at $u = c(\gamma)$ and moreover

$$\tilde{F}_{\tilde{\mathcal{T}}}(c(\gamma)) = \prod_{\beta \in \mathcal{E}_-(\mu)} (c(\gamma) - c(\beta)) \prod_{\alpha \in \mathcal{E}_+(\mu) \setminus \{\gamma\}} (c(\gamma) - c(\alpha))^{-1}. \quad (31)$$

It is known that the right hand side of (31) is equal to

$$\prod_{\alpha \in \lambda} (h_\lambda(\alpha))^{-1} \prod_{\alpha \in \mu} h_\mu(\alpha). \quad (32)$$

Define $\tilde{\mathcal{T}}$ to be the tableau of the standard 2-tableau \mathcal{T} which contains the node with number $n+1$. The assertion of the Proposition 4 follows, the only observation one has to make is that the 2-nodes (α, k) with $p_k \neq p_{n+1}$ do not contribute to (29). \square

The Theorem 2 follows, by induction on n , from the formula (15), the Proposition 3 and the Proposition 4. \square

4. Fusion procedure for the complex reflection group $G(m, 1, n+1)$

We extend the results of the previous Section to the complex reflection groups $G(m, 1, n+1)$ for all positive integers m . We skip the proofs when they are completely similar to the proofs in the preceding Section; we only indicate modifications.

4.1 Definitions

The complex reflection group $G(m, 1, n+1)$ is generated by the elements s_1, \dots, s_n and t with the defining relations (1), (2) and

$$t^m = 1. \quad (33)$$

In particular, $G(1, 1, n+1)$ is isomorphic to the symmetric group S_{n+1} and $G(2, 1, n+1)$ to B_{n+1} .

We extend the definition (4) to the generators s_1, \dots, s_n of $G(m, 1, n+1)$:

$$\mathbf{s}_i(p, p', a, a') := s_i + \frac{\delta_{p, p'}}{a - a'}, \quad i = 1, \dots, n. \quad (34)$$

The Jucys–Murphy elements for the group $G(m, 1, n+1)$ are the elements j_i , $i = 1, \dots, n+1$, and \tilde{j}_i , $i = 1, \dots, n+1$, of the group ring defined inductively by the following initial conditions and recursions:

$$j_1 = t, \quad j_{i+1} = s_i j_i s_i \quad \text{and} \quad \tilde{j}_1 = 0, \quad \tilde{j}_{i+1} = s_i \tilde{j}_i s_i + \frac{1}{m} \sum_{k=0}^{m-1} j_i^k s_i j_i^{m-k}. \quad (35)$$

For $m = 1$, that is, for S_{n+1} , $j_k = 1$, $k = 1, \dots, n+1$; the recursion formula for \tilde{j}_{i+1} reduces to $\tilde{j}_{i+1} = s_i \tilde{j}_i s_i + s_i$.

As for $m = 2$, the elements $j_i, i = 1, \dots, n+1$, and $\tilde{j}_i, i = 1, \dots, n+1$, form a maximal commutative set in $\mathbb{C}G(m, 1, n+1)$, see [20, 21]; in addition, j_i and \tilde{j}_i commute with all s_k , except s_i and s_{i-1} .

The definitions of a 2-partition, 2-node, standard 2-tableaux and hook length generalize naturally to any m ; for example, an m -partition is an m -tuple of partitions and an m -node $\alpha^{(m)}$ is a pair (α, k) , $k = 1, \dots, m$. For an m -node $\alpha^{(m)} = (\alpha, k)$ of an m -partition $\lambda^{(m)}$ such that the node α lies in the line x and the column y of the k -th diagram, we define $c(\alpha^{(m)}) := c(\alpha) = y - x$. Let $\{\xi_1, \dots, \xi_m\}$ be the set of distinct m -th roots of unity, ordered arbitrarily; we define $p(\alpha^{(m)}) := \xi_k$.

As for $m = 2$, we define

$$f_{\nu^{(m)}} := \left(\prod_{\alpha^{(m)} \in \nu^{(m)}} h_{\nu^{(m)}}(\alpha^{(m)}) \right)^{-1}, \quad (36)$$

where $h_{\nu^{(m)}}(\alpha^{(m)})$ is the hook length of $\alpha^{(m)}$ calculated in $\nu^{(m)}$.

The irreducible representations of $G(m, 1, n+1)$ are parameterized by the m -partitions of length $n+1$; for a given m -partition $\lambda^{(m)}$, elements of the semi-normal basis of the corresponding representation are indexed by the standard m -tableaux of shape $\lambda^{(m)}$. We denote by $E_{\mathcal{T}}$ the idempotent of the group ring corresponding to a standard m -tableau \mathcal{T} .

4.2 Fusion formula for idempotents of $G(m, 1, n+1)$

Let $\lambda^{(m)}$ be an m -partition of length $n+1$; fix a standard m -tableau \mathcal{T} of shape $\lambda^{(m)}$. Let \mathcal{U} be the standard m -tableau obtained by removing from \mathcal{T} the m -node containing $n+1$ and let $\mu^{(m)}$ be the shape of \mathcal{U} . Set, for $i = 1, \dots, n+1$, $p_i := p(\mathcal{T}|i)$ and $c_i := c(\mathcal{T}|i)$

The same reasoning as in Subsection 2.2 leads to the formula for $E_{\mathcal{T}}$ (cf (12)):

$$E_{\mathcal{U}} \frac{u - c_{n+1}}{u - \tilde{j}_{n+1}} \frac{v - p_{n+1}}{v - j_{n+1}} \Big|_{\substack{u = c_{n+1} \\ v = p_{n+1}}} = E_{\mathcal{T}}. \quad (37)$$

Since j_{n+1} takes values in $\{\xi_1, \dots, \xi_m\}$, the rational function $\frac{v - p_{n+1}}{v - j_{n+1}}$ is non-singular for $v = p_{n+1}$ and

$$\frac{v - p_{n+1}}{v - j_{n+1}} \Big|_{v = p_{n+1}} = \frac{1}{m} \sum_{k=0}^{m-1} p_{n+1}^{m-k} j_{n+1}^k. \quad (38)$$

An analogue of (14) is

$$\mathbf{j}_i(p) := \frac{1}{m} \sum_{k=0}^{m-1} p^{m-k} j_i^k \quad \text{for } i = 1, \dots, n+1. \quad (39)$$

For $i = 1$ we shall write $\mathbf{t}(p) := \frac{1}{m} \sum_{k=0}^{m-1} p^{m-k} t^k$ instead of $\mathbf{j}_1(p)$. Combining (37) and (38), we obtain for the idempotent $E_{\mathcal{T}}$ (cf (15))

$$E_{\mathcal{T}} = E_{\mathcal{U}} \mathbf{j}_{n+1}(p_{n+1}) \frac{u - c_{n+1}}{u - \tilde{j}_{n+1}} \Big|_{u = c_{n+1}}. \quad (40)$$

We generalize the Lemma 1 to an arbitrary positive integer m .

Lemma 5. For any integer l , $1 \leq l \leq n$, we have

- (i) $\tilde{j}_{n+1} = s_n s_{n-1} \dots s_l \tilde{j}_l s_l \dots s_{n-1} s_n + \frac{1}{m} \sum_{i=l}^n s_n \dots s_{i+1} s_i s_{i+1} \dots s_n \sum_{k=0}^{m-1} j_{n+1}^k j_i^{m-k}.$
- (ii) $\mathbf{j}_l(p) s_l \dots s_{n-1} s_n \tilde{j}_{n+1} = \mathbf{j}_l(p) \tilde{j}_l s_l \dots s_{n-1} s_n + \frac{1}{m} \sum_{i=l}^n s_l s_{l+1} \dots s_{i-1} \cdot s_{i+1} \dots s_{n-1} s_n \mathbf{j}_i(p) \sum_{k=0}^{m-1} j_{n+1}^k j_i^{m-k};$
the product $s_l s_{l+1} \dots s_{i-1}$ in the right-hand-side of (ii) is understood to be equal to 1 if $i = l$.

Proof. The proof is completely similar to the proof of the Lemma 1. \square

Let $\phi_1(v, u) := \mathbf{t}(v)$; for $k = 1, \dots, n$ define

$$\begin{aligned} \phi_{k+1}(v_1, \dots, v_k, v, u_1, \dots, u_k, u) &:= \mathbf{s}_k(v, v_k, u, u_k) \phi_k(v_1, \dots, v_{k-1}, v, u_1, \dots, u_{k-1}, u) s_k \\ &= \mathbf{s}_k(v, v_k, u, u_k) \mathbf{s}_{k-1}(v, v_{k-1}, u, u_{k-1}) \dots \mathbf{s}_1(v, v_1, u, u_1) \mathbf{t}(v) s_1 \dots s_{k-1} s_k. \end{aligned} \quad (41)$$

The formula (41) reads in the same way for any m ; only the definition of $\mathbf{t}(v)$ depends on m .

Define the following rational function with values in the group ring of $G(m, 1, n)$:

$$\Phi(v_1, \dots, v_{n+1}, u_1, \dots, u_{n+1}) := \prod_{k=0, \dots, n}^{\leftarrow} \phi_{k+1}(v_1, \dots, v_k, v_{k+1}, u_1, \dots, u_k, u_{k+1}) ; \quad (42)$$

the arrow over \prod indicates that the (non-commuting) factors are taken in the descending order.

Theorem 6. The idempotent $E_{\mathcal{T}}$ corresponding to the standard m -tableau \mathcal{T} of shape $\lambda^{(m)}$ can be obtained by the following consecutive evaluations

$$E_{\mathcal{T}} = f_{\lambda^{(m)}} \Phi(v_1, \dots, v_{n+1}, u_1, \dots, u_{n+1}) \Big|_{v_i=p_i, i=1, \dots, n+1} \Big|_{u_1=c_1} \dots \Big|_{u_n=c_n} \Big|_{u_{n+1}=c_{n+1}}. \quad (43)$$

Proof. Define

$$F_{\mathcal{T}}(u) := \frac{u - c_{n+1}}{u} \prod_{i=1}^n \frac{(u - c_i)^2}{(u - c_i)^2 - \delta_{p_i, p_{n+1}}}. \quad (44)$$

Proposition 7. We have

$$F_{\mathcal{T}}(u) \phi_{n+1}(p_1, \dots, p_n, p_{n+1}, c_1, \dots, c_n, u) E_{\mathcal{U}} = \frac{u - c_{n+1}}{u - \tilde{j}_{n+1}} \mathbf{j}_{n+1}(p_{n+1}) E_{\mathcal{U}}. \quad (45)$$

Proof. The proof follows the same lines as the proof of the Proposition 3; actually it is exactly the same until the calculation of $\mathbf{j}_l(p_{n+1}) s_l s_{l+1} \dots s_n (u - \tilde{j}_{n+1}) E_{\mathcal{U}}$ just after the formula (25). We give the modified end of the proof.

Here, we rewrite $\mathbf{j}_l(p_{n+1}) s_l s_{l+1} \dots s_n (u - \tilde{j}_{n+1}) E_{\mathcal{U}}$ using the Lemma 5, (ii):

$$\left(\mathbf{j}_l(p_{n+1}) (u - \tilde{j}_l) s_l s_{l+1} \dots s_n - \frac{1}{m} \sum_{i=l}^n s_l s_{l+1} \dots s_{i-1} \cdot s_{i+1} \dots s_{n-1} s_n \mathbf{j}_i(p_{n+1}) \sum_{k=0}^{m-1} j_{n+1}^k j_i^{m-k} \right) E_{\mathcal{U}}. \quad (46)$$

As $j_i E_{\mathcal{U}} = p_i E_{\mathcal{U}}$ for $i = 1, \dots, n$, the expression (46) is equal to

$$\left(\mathbf{j}_l(p_{n+1})(u - \tilde{j}_l) s_l s_{l+1} \dots s_n - \frac{1}{m} \sum_{i=l}^n s_l s_{l+1} \dots s_{i-1} \cdot s_{i+1} \dots s_{n-1} s_n \sum_{k=0}^{m-1} p_{n+1}^k p_i^{m-k} \mathbf{j}_{n+1}(p_i) \right) E_{\mathcal{U}} . \quad (47)$$

Since $\frac{1}{m} \sum_{k=0}^{m-1} p_{n+1}^k p_i^{m-k} = \delta_{p_i, p_{n+1}}$ we obtain that $\mathbf{j}_l(p_{n+1}) s_l s_{l+1} \dots s_n (u - \tilde{j}_{n+1}) E_{\mathcal{U}}$ equals

$$\left(\mathbf{j}_l(p_{n+1})(u - \tilde{j}_l) s_l s_{l+1} \dots s_n - \delta_{p_l, p_{n+1}} s_{l+1} \dots s_{n-1} s_n \mathbf{j}_{n+1}(p_l) \right) E_{\mathcal{U}} . \quad (48)$$

This concludes the proof. \square

The analogue of the Proposition 4 holds as well.

Proposition 8. *The rational function $F_{\mathcal{T}}(u)$, defined by (44), is regular at $u = c_{n+1}$ and moreover*

$$F_{\mathcal{T}}(c_{n+1}) = f_{\lambda^{(m)}}(f_{\mu^{(m)}})^{-1}. \quad (49)$$

Proof. The proof is completely similar to the proof of the Proposition 4. \square

Similarly to the Theorem 2, the Theorem 6 follows, using induction on n , from the formula (40), the Proposition 7 and the Proposition 8. \square

For calculations, it is sometimes useful to write the function Φ in a slightly different form. Namely, let $\tilde{\phi}_1(v, u) := 1$ and define

$$\begin{aligned} \tilde{\phi}_{k+1}(v_1, \dots, v_k, v, u_1, \dots, u_k, u) &:= \mathbf{s}_k(v, v_k, u, u_k) \tilde{\phi}_k(v_1, \dots, v_{k-1}, v, u_1, \dots, u_{k-1}, u) s_k \\ &= \mathbf{s}_k(v, v_k, u, u_k) \mathbf{s}_{k-1}(v, v_{k-1}, u, u_{k-1}) \dots \mathbf{s}_1(v, v_1, u, u_1) s_1 \dots s_{k-1} s_k , \end{aligned} \quad (50)$$

for $k = 1, \dots, n$. The elements $\tilde{\phi}_{k+1}(v_1, \dots, v_k, v, u_1, \dots, u_k, u)$ do not involve the generator t and $\Phi(v_1, \dots, v_{n+1}, u_1, \dots, u_{n+1})$, defined in (42), equals

$$\prod_{k=0, \dots, n}^{\leftarrow} \tilde{\phi}_{k+1}(v_1, \dots, v_{k+1}, u_1, \dots, u_{k+1}) \cdot \mathbf{j}_1(v_1) \mathbf{j}_2(v_2) \dots \mathbf{j}_{n+1}(v_{n+1}) . \quad (51)$$

For example, let $m = 2$; choose the order $\{1, -1\}$ on the set of square roots of 1. The primitive idempotent, corresponding to the standard 2-tableau $(\boxed{1} \boxed{3}, \boxed{2})$ reads $s_2(1 + s_1) s_2 \mathbf{j}_1(1) \mathbf{j}_2(-1) \mathbf{j}_3(1)/16$.

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